Announcements

1) We still have a mentor!

2) Official office hours

MT 10:30 - 11:30 AM

Tr 4:15 - 5:15 PM

Consequences of Triangle Inequality

Let a, b, c be real #'s

1) $|a-c| \leq |a-b| + |b-c|$

2) $\big||a|-|b|\big| \leq |a-b|$

Proofs

1) Trick: add zero

$$|a-c| = |a+0-c|$$

$$= |a+(-b+b)-c|$$

$$= |(a-b)+(b-c)|$$

apply triangle inequality

with $x = a-b$, $y = b-c$

$$\begin{array}{ccc} |\underbrace{x+y}| & \le & |x|+|y| \\ \parallel & & \parallel \\ |a-c| & \le & |a-b|+|b-c| \end{array}$$

2) on homework #2, easy!

$\square$

Completeness of the Real Numbers

Notation: The symbol $\boxed{\mathbb{R}}$ will denote all real numbers.

We regard the real numbers as a "filling in" of the rational numbers. We will be more precise on what this means in the future.

Definition (upper / lower bounds)

Let S be a subset of the real numbers.

1) $\alpha \in \mathbb{R}$ is called a lower bound for S if $\alpha \leq x \ \forall \ x \in S$.

2) $\beta \in \mathbb{R}$ is called an upper bound for S if $\beta \geq x \ \forall \ x \in S$

Note: Upper/lower bounds are never unique.

Example: Let $S = (0,1)$.

$\alpha = -5$ is a lower bound for S

So is any number less than -5

$\beta = 2$ is an upper bound for S,

so is any number greater than 2

Definition: (lub, glb) A number α is called the greatest lower bound of a set $S \subseteq \mathbb{R}$ if α is a lower bound of S and if $x \geq \alpha$, then x is not a lower bound for S.

Similarly, $\beta \in \mathbb{R}$ is called the least upper bound of $S \subseteq \mathbb{R}$ if β is an upper bound of S and if $y \leq \beta$, then y is not an upper bound for S.

Back to our example with $S = (0,1)$, $\alpha = 0$ is the greatest lower bound of S and $\beta = 1$ is the least upper bound.

Same greatest lower bound and least upper bound for

$$S = [0,1],\ [0,1),\ (0,1]$$

Note: it is not a consequence of the definition that that the greatest lower bound or the least upper bound be in the given set S.

Note also that upper or lower bounds need not exist

Examples: $\mathbb{Z}$, $\mathbb{Q}$, or $\mathbb{R}$ have no upper or lower bounds

$\mathbb{N}$ has no upper bound, but the greatest lower bound is $\alpha = 1$.

Alternate Notation

We sometimes call the greatest lower bound of $S \subseteq \mathbb{R}$ the infimum of S and write $\inf(S)$. Similarly, the least upper bound is sometimes called the supremum of S, and is written $\sup(S)$

Also $\mathrm{glb}(S)$ or $\mathrm{lub}(S)$ are sometimes used for greatest lower or least upper bounds

Examples:

1) Recall from Calc 2 that

$$e = \sum_{n=0}^{\infty} \frac{1}{n!}$$

$n! = n(n-1) \cdots 2 \cdot 1$

$0! = 1$

Think of the set S as all "partial sums" starting from $n=0$.

$$S = \left\{ 1, 2, \frac{5}{2}, \frac{5}{2} + \frac{1}{6}, \frac{5}{2} + \frac{1}{6} + \frac{1}{24}, \cdots \right\}$$

Each subsequent term in S is larger than the last, and all elements of S are rational!

The number e is the least upper bound for S.

Note that e is irrational (Euler?) and even transcendental

So the sup of a set of rational numbers need not be rational!

Completeness Axiom for $\mathbb{R}$

Let $S \subseteq \mathbb{R}$ be bounded from above. Then S has a least upper bound $\beta \in \mathbb{R}$. Similarly, if S is bounded below, then S has a greatest lower bound $\alpha \in \mathbb{R}$.

Example 1 Let p be a prime number and let

$$S = \{ x \in \mathbb{Q} : 0 \leq x^2 < p \}$$

This set S is definitely bounded above, the least upper bound is $\sqrt{p}$, which we showed is not a rational number

Proposition (characterization of sup/inf) Let $S \subseteq \mathbb{R}$.

Then $\beta \in \mathbb{R}$ is the least upper bound of S if and only if β is an upper bound of S and for every $\varepsilon > 0$, $\exists\, x \in S$ with $\beta - \varepsilon < x$.

Before the proof. In this class, "ε" is a greek letter that will always refer to positive real numbers. In most examples,

ε = "small" positive real number

The translation of the proposition is that $\beta = \sup(S)$ if and only if you can get "as close as you like" to β with elements in S that are smaller than β

Next time: Section 1.4.